EMS
**European Mathematical
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**Committee on
Mathematics Education**

**Niveaux de référence pour l'enseignement
des mathématiques en Europe**

**Reference levels in School Mathematics
Education in Europe**

National Presentation
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1. Teaching of mathematics at compulsory school in Denmark

Compulsory school ('Folkeskolen') in Denmark start by the age of 7 and consists of 9 grades (years of study). It comprizes a connected curriculum for primary school and lower secondary school. Mathematics is part of the curriculum throughout all grades and next to Danish it is the largest subject in compulsory school. Finishing compulsory school, the pupils are expected to know the four basic operations in calculations with (real) numbers, elements of arithmetic, basic geometric figures and formulas for the area of the most basic figures in the plane. A main problem with teaching of mathematics in compulsory school in Denmark is that less than 15% of the teachers have mathematics at the highest level from the teacher training colleges. In reality the remaining 85% only have their own background in mathematics from primary and lower secondary school as background for their work as teachers of mathematics.

In later years, there has been much public discussion whether pupils actually master such fundamental topics as mentioned above after finishing compulsory school. In particular, it has been argued that even pupils with good intellectual abilities do not master calculations with fractions and power operations with powers of 10. The result of this being that they do not understand symbolic algebraic manipulations. Proofs are also mostly left out in connection with geometrical arguments, such that pupils have a very weak and shaky feeling for logical

thinking in connection with proofs when they enter upper secondary school. The following quote from a recent mathematics book from compulsory school in connection with the proof that the sum of angles in a triangle is 180 degrees may serve as an illustration. The 'proof' goes as follows:

"Draw a number of different triangles and measure the angles. As long as you cannot find a triangle with another sum of the angles than 180 degrees, we accept that the theorem is true."

Very recently, steps have been taken by the Ministry of Education to improve the situation, and in new regulations to be installed August 2001, it is required that certain explicitly formulated, fundamental mathematical skills are taught in compulsory school.

Among the topics singled out for general comparison among countries in this European study of reference levels in mathematics, we shall only briefly comment on a few topics from compulsory (primary and lower secondary) school in Denmark:

- The Pythagorean theorem is stated but not proved in compulsory school.
- The general second degree equation is not dealt with in compulsory school, exept for solution of special cases by inspection.
- The general notion of percentages is not fully developed in compulsory school.
- The use of mathematical symbols and mathematical precision is kept at a minimum in compulsory school; much talking around things, guessing and 'experiments' without explanations, few formulas.

2. Teaching of mathematics at secondary school in Denmark

About 53% of all children in Denmark continue from compulsory school to attend a general secondary school education, about 41% continue in vocational colleges (technical or commercial) or basic social and health care educations, and about 6% leave or drop out of the education system after compulsory school.

The general upper secondary school education in Denmark is divided into the Gymnasium (see next section), which is a 3 years course leading to Secondary School Leaving Examination ('studentereksamen'), and HF (Higher Prepatory Examination), which is a 2 years course.

At the vocational colleges and in the basic social and health care educations, the amount of mathematics is strongly varying and dependent on the profession. Presently, many ressources are devoted to develop relevant mathematics courses in connection with upgrading of adults in order for them to obtain mathematical skills which have turned out to be necessary for them in their professions.

By law, the Danish secondary school teachers in the traditional Gymnasium and the HF have a masters degree (M.Sc.) from a university and have a comprehensive understanding of mathematics. Due to the above mentioned asserted weaknesses in the preparations in mathematics from primary school, teachers in secondary schools now often have to teach pupils mathematical skills which was not a problem before for pupils entering secondary school. It has to be said that a much larger fraction of children than before enter secondary

school, a fact that may explain some of the problems with lacking mathematical skills in the transition from primary to secondary school, but certainly not *all* the problems.

The problems with lacking mathematical skills are now also felt at university level, and most problematically in studies where mathematics is not the main field of study but an important necessary subject for understanding the field of study. Quite recently, however, the lack of mathematical skills have also shown up in an even more unpleasant context, namely among mathematics candidates from universities in pedagogical training for becoming secondary school teachers. An alarming number of these trainees for secondary school teaching have difficulties with elementary mathematical skills, something which was unheard of just a few years ago.

Recently, the Danish Ministry of Education has appointed a commission to investigate what sort of *competences*, as distinct from a specific description of a curriculum, students should have obtained in the transition from one level to the next in the educational system.

3. The Danish Gymnasium
Upper secondary school education in Denmark earlier took place exclusively in the Gymnasium. During the last 30 years several new types of upper secondary school education have developed. All these types of educations, including the traditional Gymnasium, are called *gymnasiale uddannelser* (gymnasium level educations). One of the new gymnasium level educations is linked to trade and economics: HHX (Higher Trade Examination), created 1972, and another to engineering: HTX (Higher Technical Examination), created 1982. Since the concern here is mathematics, and the level of mathematics is fixed within the same three levels A, B and C at all the gymnasium level educations, we shall often in the following use the word *gymnasium* for upper secondary school teaching in mathematics in Denmark. But it should be understood that the word gymnasium then covers several types of upper secondary educations, which can be rather different when all subjects in the particular education are included. It should also be emphasized that a level in real life is never a 'point' but an 'interval', since it may be rather different what actually goes on in the various types of educations.

4. Short history of upper secondary school teaching in Denmark
In the second half of the nineteenth century, secondary school education in Denmark developed into the above mentioned Gymnasium, formalized by a major school reform in1903. The traditions for the Gymnasium were laid in the old latin school system streching back in time for many centuries to teaching in connection with abbeys and churches. The emphasis in the latin school system was on general education (German: "bildung"), and teaching of classical languages (latin and greek) and Antiquity were essential elements. The purpose was to equip the elite with culture and to prepare it for further studies directed towards Civil Service.

The increasing importance of the natural sciences, in particular the needs for well qualified engineers, led to a more systematic teaching of subjects from the natural sciences and to the creation of a new mathematics-science line in the latin school in 1871 alongside with the classical languages line. At the school reform in 1903 a third line was established in the Gymnasium with emphasis on modern languages. These three lines: classical languages, mathematics-science, modern languages continued up to 1971, where the mathematics-science line was divided into three branches: mathematics-physics, mathematics-biology, mathematics-

social sciences, the names reflecting the emphasis. A similar division into branches happened with the linguistic lines.

As clearly indicated by the names, mathematics stands out as the common and central subject in all branches of the mathematics-science line. Similarly, English became the central subject in all branches of the linguistic lines. Thereby it was now established that secondary school in Denmark consists of a Linguistic Gymnasium and a Mathematics Gymnasium. By a later reform in 1987, this was formally recognised and gymnasium in Denmark now in reality is divided into the *Linguistic Gymnasium* and the *Mathematics Gymnasium.* It is noteworthy that by every single reform of the Gymnasium since 1871, the number of lessons for mathematics has been cut.

Since mathematics is a prerequisite for an increasing number of further educations, there has throughout the years been a discussion of a place for mathematics also in the Linguistic Gymnasium. Using the terminology of the three levels in mathematics A, B and C, to be described below, it is now an option for students in the Linguistic Gymnasium to obtain level C in mathematics, and around 10% of the students even obtain level B in mathematics.

The earlier mentioned 2 years upper secondary school education HF, was introduced in the late 1960s. From the outset, the intention was to bring adults who had left the educational system right after compulsory school back for further education. And still, HF is above all directed at mature young people and adults who have left the educational system but now wishes to prepare themselves for a higher education, e.g. as primary school teachers. All students from HF obtain level C in mathematics, and have level B as an option.

It may also be appropriate to remark, that the new types of gymnasium level educations in Denmark, linked to engineering, respectively trade and economics, somehow reflects the former mathematics-physics, mathematics-biology, mathematics-social sciences and linguistic-social sciences branches in the Gymnasium. In reality, HHX is divided into a Linguistic HHX and a Mathematics HHX, whereas the HTX closely corresponds to the Mathematics Gymnasium, but with a technical and information technological toning.

As you can see from the above, educational reforms at secondary school level in Denmark has always been centred around finding a balance between a uniform secondary school educational system and a diversity of secondary school educations reflecting a number of future specializations. At the beginning of the twentyfirst century, there are strong advocates for a uniform structure of the Gymnasium but with a huge number of optional subjects. There are, however, also strong forces working for a more close interaction between subjects, and for each of the types of gymnasium level education to work out specific and different profiles. It is strange, and may be a rather unique thing in the Danish system compared to the rest of Europe, that presently there is no formal connection between mathematics and the natural sciences in the Gymnasium.

At the same time, the importance of mathematics and the natural sciences for modern society is verbally recognised by almost all Danish politicians. How mathematics will find its place in a gymnasium with a high degree of free choice among subjects is, of course, unclear. Presently, mathematics is a fairly popular subject to choose in the Gymnasium, probably not least based

on mathematics as a prerequisite for further education, but also to some extent based on mathematical interest among students.

5. Three main levels of mathematics
Mathematics at the upper secondary school level in Denmark is taught at three different levels: A, B and C, to be described below.

We begin by describing the contents of the lowest level C, which is a proper subset of the contents of level B, which is again a proper subset of the contents of the highest level A.

5.1 Level C of mathematics in upper secondary school teaching in Denmark
At this basic level, students are taught:

- *Numbers:* Various representations of numbers are treated, including fractions, decimal numbers, exponential representation as well as the representation of numbers in computers.
- *Functions:* Dependent and independent variables, the difference between a function and a particular function value, various ways of defining functions (tables, algorithms, and graphs).
- *Algebraic equations*: Solution of systems of linear equations, and solution formula for equations of second degree.
- *Plane geometry:* Altitude, bisector, median, perpendicular bisector are introduced, and the simple geometrical properties of them are discussed. Congruence of and similarity of triangles, and especially sine, cosine and tangent of the angles of a right-angled triangle.
- *Analytical geometry:* Study of the connection between equations and geometrical objects, emphasis on lines and the connection with systems of linear equations.
- *Basic statistics:* Fractiles, histograms.

5.2 Level B of mathematics in upper secondary school teaching in Denmark
Many of the topics mentioned under level C are further developed. Main new topics are:

- *Differential calculus:* The concepts of limit and continuity are treated to the extend deemed necessary for understanding the concept of a derivative. The instruction includes rules for calculating limits and continuity as well as examples of proofs of the differential quotient of simple functions, including trigonometric functions. Methods for graphic and numerical differentiation are discussed. The instruction includes rules for the differentiation of sum, difference, product, quotient and composite function; proofs are given in some cases, e.g. the product rule.
- *Statistic and probability:* Probability spaces are dealt with as models for random experiments, and examples are given of symmetrical and nonsymmetrical probability spaces. The concepts of independent occurrence and conditional probability, including Bayes' theorem. The formula for the binomial coefficients is derived, but the actual combinatorial methods are only dealt with to an extent necessary for understanding binomial distribution and hypergeometrical distribution.

5.3 Level A of mathematics in upper secondary school teaching in Denmark

Many of the topics mentioned under levels C and B are further developed. The main new topics are:

- *Integral calculus:* Selected aspects of integral calculus must be given a concentrated and rigorous treatment, and in this connection proofs are carried out for selected significant theorems. Work is done with the integral as the limit of sums as well as with the connection between the integral and primitive functions. Calculation of area and volume. Exact and numerical methods for calculating integrals must be treated, including partial integration, integration by substitution, as well as the use of integral tables.
- *Differential equations:* The notion of differential equations as mathematical models must be discussed, illustrating how the use of infinitesimal considerations leads to the construction of differential equations. No general theory for the solving of differential equations is required, but some special types, e.g. separation of variables, must be included in the instruction.
- *Geometry and vectors:* Vectors are introduced as a useful concept in geometry. Geometry in space including planes and the surface of the sphere. Conic sections are dealt with geometrically and analytically, but an actual treatment of the geometrical properties of conic sections is not required.

For all levels, and most explicitly for the highest level A, the teaching of mathematics at secondary school in Denmark should address the following three aspects of mathematics:

The historical aspect: The students must acquire a knowledge of elements of the history of mathematics and of mathematics in a cultural and social context.

The model aspect: The students must acquire a knowledge of the structure of mathematical models as representations of reality and an impression of the possibilities and limitations of applying them, and also the students must become able to carry out on their own a modelling process in simple situations.

The inner structure of mathematics: The students must acquire an understanding of the models of thinking and the methods characteristic of mathematics and an understanding of how they affect the development and structure of mathematical areas.

The students must be exposed to various presentations of mathematics. Besides textbooks, notes, etc., the students must become acquainted with texts on mathematics and/or texts including the application of mathematics. Altogether 500—700 pages are read for the highest level, depending on the teaching material chosen.

Denmark has a strong tradition for secondary school teachers writing teaching material, and there are several competing mathematics book systems for secondary school fulfilling the requirements set by the central authorities in the Ministry of Education. Many of the mathematics book systems are of high quality.

Graphical pocket calculators are compulsory for the teaching of mathematics at secondary school.

Fairly advanced mathematical programs are already being used in connection with experiments of new ways of teaching mathematics. Eventually, and many believe very soon, advanced mathematical programs will be an integrated and natural part of the teaching of mathematics at secondary school. How that will effect the teaching of mathematics can only be a guess at the moment.

The Danish system of assessment of students performances have strong traditions involving written as well as oral examinations. In the oral examinations in mathematics, proofs played earlier a significant role and it still does at the secondary school level. The reduced training in abstract thinking, and maybe even the dismay of abstract ideas, with which pupils leave primary school, has, however, affected the possibilities for presenting mathematical proofs at secondary school.

Oral examinations may in the future become more and more important in order to control that students have obtained the formal qualifications in the education in question since pocket calculators, computers and advanced mathematical programs make it difficult to perform written tests of rutine skills. For the same reason, written tests without any aids (only pencil and paper allowed) have again been introduced as part of the written examinations in mathematics at secondary school level; less than 40 years ago all written examinations in the Gymnasium were without aids.

6. Short statistics of upper secondary school teaching in Denmark
Gymnasium level education in Denmark is a 3 years long education.
There are the following types of gymnasium level educations:

- Gymnasium, mathematics
- Gymnasium, linguistic
- HHX, mathematics
- HHX, linguistic
- HTX (mathematics)

In addition to the above educations, there is also a 2 years upper secondary school eduction, HF (Higher Prepatory Examamination), mostly preparing for shorter further educations.

In Table 1 we give the approximate percentage distribution among the various upper secondary school educations for students entering these educations year 2000.

Type of education	Gymn. Math.	Gymn. Ling.	HHX Math.	HHX Ling.	HTX Math.	HF
Percentage	31 %	23 %	5 %	16 %	8 %	17 %

Table 1

The three years of education in the Gymnasium is reflected in the three levels in mathematics, where level C corresponds to one year of mathematics, level B to two years, and level A to three years of mathematics. In the gymnasium level educations with mathematics, the lowest

level is B, so that all students from these upper secondary school educations have had at least two years of mathematics.

In Table 2 we show the levels of mathematics in all the types of upper secondary school educations.

Type of higher secondary education	Compulsory level of mathematics	Optional level of mathematics
Gymnasium, Math.	B	A
Gymnasium, Ling.		C, B
HHX, Math.	B	A
HHX, Ling.		C
HTX	B	A
HF	C	B

Table 2

In Table 3 we give the number of lessons, each of 60 minutes duration, for each of the three years in the Mathematics Gymnasium:

Year	Number of lessons
1 year	132
2 year	135
3 year	127

Table 3

Table 4 show the number of students that began upper secondary school education in 1998.

Secondary School	Number of students Starting 1998
Mathematics Gymnasium	10.529
Linguistic Gymnasium	8.172
HTX, HHX, HF	15.545

Table 4

The students that started in the Mathematics Gymnasium in 1998 finished gymnasium year 2000. Approximately 15 % of the students leave gymnasium without finishing in due course. Many of the students leaving the Gymnasium change, however, to the shorter secondary school education HF and complete that education.

Of the 10.529 students entering Mathematics Gymnasium in 1998 we have Table 5 below for the level obtained in mathematics among the students completing gymnasium in 2000.

Finishing level	Total number of students
B	App. 1. 400
A	App. 7. 600

Table 5

Mathematics at level A is a requirement for many university studies.

Source:

Statistical information in the above report has been extracted from information given on the home page of the Danish Ministry of Education: http://www.uvm.dk